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Distances Realized by Sets Covering the Plane*

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A proof is given of the (known) result that, if real n-dimensional Euclidean space R^n is covered by any $n + 1$ sets, then at least one of these sets is such that each distance d ($0 < d < \infty$) is realized as the distance between two points of the set. In particular, this result holds if the plane is covered by three sets; but it does not necessarily hold if the plane is covered by six sets. If each set in a covering of the plane fails to realize the *same* distance d, say $d = 1$, and if the sets are either closed or simultaneously divisible into regions (in a sense to be made precise), then at least six sets are needed and seven suffice, and the number of closed sets needed is at least as great as the number simultaneously divisible into regions.

1. INTRODUCTION

At the end of his monthly column in the October 1960 issue of *Scientific American* (p. 180), Martin Gardner asked how many colors are needed to color the plane in such a way that no two points that are unit distance apart have the same color. He quoted L. Moser as saying, what is in any case easy to prove, that four colors are necessary and seven sufficient. This is true whether one allows colorings in which each point of the plane is colored independently, or whether one allows only colorings that can be obtained by drawing a map in the plane and coloring the regions of the map. Following the discussion on pp. 24–25 of Hadwiger, Debrunner, and Klee [5] (hereafter referred to as HDK), we may formalize this problem, and at the same time generalize it, as follows. What is the minimum number of sets with which we can cover the plane in such a way that

(a) for each set A there is some distance d_A that is not realized as the distance between any two points of the set A; or

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(b) there is some distance d, say $d = 1$, that is not realized as the distance between any two points of the same set? Here the sets may be closed (C), measurable (M), arbitrary (A), or simultaneously divisible into regions (R) (in a sense to be made precise in Section 3 below, the regions being the regions of a map), and we write X(x) (X = C, M, A or R; x = a or b) for the minimum number of sets of type X that are needed to cover the plane in such a way that (x) is satisfied. If the sets are disjoint we frequently regard them as being "colored," in the spirit of Martin Gardner's question.

We may extend these problems by specifying that the sets should be similar, in (a), or congruent, in (b). These restrictions have more force if we insist that the sets should be similarly situated (with respect to rotations), and/or that the sets should be disjoint (or, in the case of closed sets, that their interiors should be disjoint). In problem (a) we may restrict the number of different "missing distances" to 2, 3, etc. ((b) being the restriction to only one "missing distance"), or we may actually specify what these distances should be. We may also generalize the problems from the plane, R^2, to R^n. These extensions will be largely ignored in what follows, although Theorem 3 is the generalization of Theorem 2 to R^n.

L. Moser has varied the question by asking how "dense" a single plane measurable set can be if it realizes no unit distance. Croft [2] observes that the density δ must satisfy $\delta \leqslant \frac{2}{7} \simeq 0.2857$, and exhibits a set with $\delta \simeq 0.2293$. Proposition 59 of HDK shows that C(a) $\geqslant 4$, and this observation of Croft's shows that M(a) $\geqslant \frac{7}{2}$ and so M(a) $\geqslant 4$. (However, $0.2293 < \frac{1}{4}$, and, so far as I know, no one has yet exhibited a set that could be a possible condidate for one of four measurable sets covering the plane and satisfying (a).) In Theorem 2 below we shall prove that A(a) $\geqslant 4$. This result has also been obtained by Raiskii [8], who indeed proves the corresponding result for general R^n, which is Theorem 3 of the present paper. (My proofs of these results, obtained before I was aware of Raiskii's work, are slightly longer than Raiskii's, but they appear to be quite different and are perhaps of some interest in their own right.)

It is clear that A(b) $\geqslant 4$. (A simple demonstration of this is included in L. Moser and W. Moser's solution of their interesting Problem 10 in [7], and is repeated by Croft [2].) The proof of Proposition 60 of HDK shows that C(b) $\geqslant 6$, without the additional restriction (stated there) that the sets should be congruent. We shall prove in Theorem 6 below that R(b) $\geqslant 6$, and in Theorem 5 that C(x) $\geqslant$ R(x) (x = a or b). At the upper end, the example on page 24 of HDK shows that X(x) $\leqslant 7$ for all X and x, and we shall show in Theorem 4 below that X(a) $\leqslant 6$ for all X. So the position at the end of this paper will be:

$$\mathrm{C(a), R(a), M(a), and\ A(a)} = 4, 5, \text{ or } 6;$$
$$\mathrm{C(b)\ and\ R(b)} = 6 \text{ or } 7;$$
$$\mathrm{M(b)\ and\ A(b)} = 4, 5, 6 \text{ or } 7;$$
$$\mathrm{C(a)} \geqslant \mathrm{R(a)} \geqslant \mathrm{A(a)};\ \mathrm{C(a)} \geqslant \mathrm{M(a)} \geqslant \mathrm{A(a)};$$
$$\mathrm{C(b)} \geqslant \mathrm{R(b)} \geqslant \mathrm{A(b)};\ \mathrm{C(b)} \geqslant \mathrm{M(b)} \geqslant \mathrm{A(b)}.$$

It is a long-standing problem of Erdös whether or not A(b) = 4. In view of the powerful graph-coloring theorem of de Bruijn and Erdös [1], to prove A(b) = 4, it would suffice to prove that any *finite* set of points in the plane can be colored with four colors so that no two points that are unit distance apart have the same color. (This theorem of de Bruijn and Erdös assumes the axiom of choice—but there seems little hope of determining A(b) without it.) This is equivalent to proving, in the language of Erdös, Harary, and Tutte [3], that every finite graph of dimension 2 has chromatic number at most 4. (The number A(b) is referred to in [3] as the *chromatic number* of the plane.)

We conclude this introductory section with a very simple result of a similar type.

THEOREM 1. *The rational points of the plane can be colored with only two colors in such a way that no two points that are unit distance apart have the same color.*

Proof. If p/q and r/s are rationals in their lowest terms such that $(p/q)^2 + (r/s)^2 = 1$, i.e. $p^2s^2 + r^2q^2 = q^2s^2$, then exactly one of p and r is even and the other, and q and s, are odd. So if we define an equivalence relation on the rational points of the plane by writing $(a, b) \sim (c, d)$ if and only if $a - c$ and $b - d$ both have odd denominators when written in their lowest terms (including the possibility that one or both are integers), then two rational points that are unit distance apart must be from the same equivalence class. So it suffices to color a single equivalence class in the required way (since all the equivalence classes are translations of each other). We do this for the class containing the origin by assigning one color to points of the form $(o/o, o/o)$ and $(e/o, e/o)$ and the other to points of the form $(o/o, e/o)$ and $(e/o, o/o)$ (where o stands for an odd number and e for an even number). This completes the proof of Theorem 1. ∎

This coloring can be extended to a wide class of quadratic irrational points (although clearly not to all such points, in view of the unit equilateral triangle), but of course we are still only dealing with a countable set of points, with measure zero.

2. PROBLEM (a)

We start with a simple proof of Raiskii's result for the plane.

THEOREM 2. *If the plane is covered by any three sets, then at least one of the sets is such that its pairs of points realize all distances* d $(0 < d < \infty)$.

Proof. Let the three sets be A, B, and C and suppose, in contradiction to the statement of the theorem, that they fail to realize distances d_A, d_B and d_C, respectively, where we may suppose without loss of generality that the three sets are disjoint, that each contains points of the plane, and that $d_A \leqslant d_B \leqslant d_C$. For future reference we divide the proof into two steps:

Step 1. Let P_C be any point of the plane in set C, and let S_C be a circle with radius d_C and center P_C. S_C is covered by sets A and B. Now, two points x and y in S_C that are distance d_A apart cannot both be in A, nor both in B: for the two points x' and y' in S_C, distant d_B from x and y, respectively, in the positive sense round S_C, are distance d_A apart, and if x and y were both in B then x' and y' would both be in A, which is impossible. So one of x and y is in A and the other is in B. If P is a point distant d_{AC} from P_C, where $d_{AC} := (d_C^2 - \frac{1}{4}d_A^2)^{1/2} \pm d_A \sqrt{3}/2$, then the two points x and y in S_C distant d_A from P are also distant d_A from each other, and so one of them is in A.[1] Thus P is not in A, which proves that the distances d_{AC} cannot be realized as the distance between a point of the plane in C and one in A. We choose the larger distance d_{AC}, so that $d_{AC} > d_C \geqslant d_B$.

Step 2. Now let $d := (d_{AC}^2 - \frac{1}{4}d_B^2)^{1/2} \pm d_B \sqrt{3}/2$. These distances cannot be realized as the distance between a point in C and one in B: for if S_C' is a circle of radius d_{AC} centered on a point in C, then S_C' is covered by the sets B and C, and we can repeat the above argument with d_B in place of d_A. But, equally, d cannot be the distance between a point in A and one in B: for, if S_A' is a circle of the same radius d_{AC} centered on a point in A, then S_A' is covered by the sets A and B, and the same argument goes through again. So a circle of radius d centered on a point in B must lie entirely in B, and this means that the whole plane must be in B. This contradiction completes the proof of Theorem 2. ▌

[1] Throughout the paper the symbol $:=$ or $=:$ indicates that the equation in which it occurs acts as the definition of (some part of) the expression on the same side of the equality sign as the colon. The symbol ▌ denotes the end (or absence) of a proof.

Hadwiger [4] proved that, if n-dimensional Euclidean space R^n is covered by $n + 1$ closed sets, then at least one of the sets is such that its pairs of points realize all distances d ($0 < d < \infty$). Larman [6] proved this for arbitrary sets with "almost all" in place of "all" distances d. Finally, Raiskii [8] proved it for arbitrary sets and "all" distances d. We now provide a new proof of this result, along the lines of the proof of Theorem 2.

THEOREM 3. *If n-dimensional Euclidean space R^n is covered by any $n + 1$ sets ($n = 2, 3,\ldots$), then at least one of the sets is such that its pairs of points realize all distances d ($0 < d < \infty$).*

Proof. Let the sets be $A_0, A_1, \ldots, A_n$, and suppose that A_i fails to realize distance d_i ($i = 0, 1,\ldots, n$), where as before we may suppose without loss of generality that the sets are disjoint and that $d_0 \leqslant d_1 \leqslant \cdots \leqslant d_n$.

Let P_n be a point of R^n in A_n or, if no such point exists, any point of R^n. Let S_{n-1} be the set of points of R^n at distance d_n from P_n. S_{n-1} is a sphere in R^n, covered by sets $A_0, A_1, \ldots, A_{n-1}$, of radius d_n.

Let P_{n-1} be a point of S_{n-1} in A_{n-1} or, if no such point exists, any point of S_{n-1}. Let S_{n-2} be the set of points of S_{n-1} at distance d_{n-1} from P_{n-1}. S_{n-2} is a sphere in R^{n-1}, covered by sets $A_0, A_1, \ldots, A_{n-2}$, of radius r_{n-2} (say), where, as noted by Larman [6]:

$$(r_{n-2}/d_{n-1})^2 + (d_{n-1}/2d_n)^2 = 1.$$

Thus

$$r_{n-2} = d_{n-1}\{1 - (d_{n-1}/2d_n)^2\}^{1/2} \geqslant d_{n-1}\sqrt{\tfrac{3}{4}} > d_{n-1}/\sqrt{2}.$$

We may proceed in this way, constructing spheres $S_{n-3}, S_{n-4}, \ldots$, until we obtain a sphere S_2 in R^3 that is covered by sets A_0, A_1, and A_2 and has radius r_2. At each stage we have

$$r_{i-1} = d_i\{1 - (d_i/2r_i)^2\}^{1/2} > d_i/\sqrt{2} \qquad \text{if} \quad r_i > d_{i+1}/\sqrt{2} \geqslant d_i/\sqrt{2},$$

and so we have $r_2 > d_3/\sqrt{2} \geqslant d_2/\sqrt{2}$.

Let us write A, B, C, d_A, d_B and d_C instead of A_0, A_1, A_2, d_0, d_1 and d_2. It remains to show that we can repeat the argument of Theorem 2, not in the plane, but in (the surface of) a sphere S_2 in R^3 of radius r ($=r_2$) $> d_C/\sqrt{2}$. As before, we may suppose without loss of generality that each of A, B, and C contains points of S_2. We could carry out the argument of Theorem 2 exactly (with suitable reinterpretation of the distances d_{AC} and d, which will be presupposed throughout the following discussion), if, for example, $d_C < r$. But the bound $d_C < r\sqrt{2}$ is critical in that, if we could have $d_A = d_B = d_C = r\sqrt{2}$, then this method would fail completely.

As it is, we can complete step 1 of the proof, for, in S_2, S_C has radius r_C (say), where

$$r_C = d_C\{1 - (d_C/2r)^2\}^{1/2} > d_C/\sqrt{2}.$$

But, in step 2, the most we can say about the radius r_{AC} of S_C' and S_A' is that it is strictly positive, and if $r_{AC} < \frac{1}{2}d_C$ then this step of the proof fails. So we have to modify the proof to get round this. We may suppose that $r_{AC} < \frac{1}{2}d_C$, since otherwise the original proof goes through as before (with the smaller of the two distances d, in case the larger is equal to $2r$). We may also suppose that $d_A < d_C$, since if $d_A = d_B = d_C < r\sqrt{2}$ we can quickly derive a contradiction exactly as in the plane. We distinguish two cases.

Case 1. $d_B \leqslant d_C/\sqrt{2}$ $(<r)$. In this case it is not difficult to see that $r_{AC} > \frac{1}{2}r > \frac{1}{2}d_B$, and so we simply insert an extra step in the proof, step 1A, as follows. Consider the set of points S_A' in S_2 which are distant d_{AC} from a point P_A of S_2 in A. S_A' is a circle of radius $r_{AC} > \frac{1}{2}d_B \geqslant \frac{1}{2}d_A$, and it is covered by sets A and B. We may therefore repeat the argument of step 1 to form a distance $d_{AB} < d_{AC}$ which cannot be realized between a point of A and a point of B. But since $r_{AC} < \frac{1}{2}d_C < r/\sqrt{2}$ we must certainly have $d_A > 2r\sin\frac{1}{8}\pi$, and, since $d_B \geqslant d_A$ and $r_{AC} > \frac{1}{2}r$, this suffices to ensure that the radius r_{AB}, of the circle of points of S_2 distant d_{AB} from a fixed point of S_2, satisfies $r_{AB} > r/\sqrt{2} > \frac{1}{2}d_C$. So we may now repeat the argument of step 2 of Theorem 2 with B and C interchanged throughout.

Case 2. $d_B > d_C/\sqrt{2}$. In this case, in step 1 of the old proof we select the smaller distance d_{AC} instead of the larger, where $d_{AC} > 0$ since $d_A < d_C$, and

$$d_{AC} < r_{AC}\sqrt{2} < d_C/\sqrt{2} < d_B.$$

Now choose a point P_B of S_2 in B, and consider the set of points of S_2 at distance d_B from P_B. This forms a circle S_B of radius $r_B > d_B/\sqrt{2} > \frac{1}{2}d_C$, and so we may use the usual argument to deduce that two points of S_B separated by distance d_A or d_C must consist of one point in A and one in C. We also know that two points of S_B separated by distance d_{AC} must be in the same set. Using these facts we may construct a distance $d' \leqslant d_{AC}$ such that two points x and y of S_B separated by distance d' must consist of one point in A and one in C, and now a point in S_2 distant d_{AC} from x and y must be in B. Thus we can construct a distance d as before such that any point of S_2 at distance d from a point in B is also in B, and this contradiction completes the proof of Theorem 3. ∎

We now return to the plane, and prove the following upper bound:

THEOREM 4. $X(a) \leqslant 6$, *for* $X = C, M, R$ *or* A.

Proof. We have to construct a covering of the plane by six sets such that for each set there is some distance that is not realized as the distance between two points of that set. It is not difficult to do this with sets that are not closed, as was shown by Raiskii [8]. His example is obtained from the "densest possible" (regular triangular) packing of the plane by disjoint open unit discs. Four colors suffice to color the discs in such a way that no color realizes distance 2, and two colors suffice for the remaining curvilinear "triangles" with no color realizing unit distance (except that some of the vertices of the "triangles" have to be given the color of an adjoining disc—it is not difficult to find a suitable rule). However, in this and similar coverings based on the regular triangular, square, and hexagonal lattices the distances are critical, in that each set fails to realize exactly one distance, and so we cannot simply replace each set by its closure. In fact I do not know of such a covering by closed sets in which there are only two "missing distances."

We can, however, construct such a closed covering with three "missing distances"—and, indeed, one in which the six sets are all similar, although not similarly situated—as follows. We start with the infinite regular triangular lattice in the plane, composed of equilateral triangles with unit side. Centered on each vertex of each triangle (i.e., on each lattice point) we place an open disc of radius $\frac{1}{4}\sqrt{3}$: three colors suffice for these discs, with no color realizing distance $\frac{1}{2}\sqrt{3}$. Centered on the centroid of each triangle we place an open disc of radius $\frac{1}{4}$: two colors suffice for these discs, with no color realizing distance $\frac{1}{2}$. Finally, centered on the midpoint of each side of each unit triangle we place an open disc of radius $\frac{1}{8}$: These discs can all be given the same color without realizing distance $\frac{1}{4}$. This of course is an open covering: to convert it into a closed covering, we must replace each open disc of radius r by a concentric closed disc of radius $r - \epsilon$ for some sufficiently small ϵ, say $\epsilon = 10^{-2}$, or of radius $r(1 - \epsilon)$ if we wish to preserve the similarity of the sets. No doubt one could give a theoretical proof that these sets do indeed cover the plane, but personally I find an accurate scale drawing more convincing. Indeed, there is so much latitude that we could almost certainly shrink the smallest discs until the sixth set failed to realize distance $\frac{1}{4}(1 + \sqrt{3}) \simeq 0.68$; but I have not managed to make two of the three "missing distances" equal in this way. Nevertheless, the proof of Theorem 4 is complete. ∎

3. Problem (b); Covering by Sets Simultaneously Divisible into Regions

When we wish to construct a covering of the plane satisfying (a) or (b), it is natural to try to do this by drawing a suitable map in the plane and coloring the regions of the map. Theorem 5 below shows that our intuition is to some extent justified, in that, if we are looking for examples of closed coverings, we may indeed restrict our attention to examples of this type. We now formalize this concept.

Let G be a proper embedding of a graph in R^2; so G consists of a set of points of R^2, called *vertices*, together with a set of Jordan arcs, called *edges*, such that each edge joins two vertices, passes through no other vertex and meets no other edge except at an end point (vertex). Suppose that in each bounded region of the plane there are at most finitely many vertices, and points of at most finitely many different edges. (This finiteness restriction arises naturally in Theorem 5: we do not in fact need the full force of it in Theorem 6, but it seems simpler to impose it than to list a set of alternative axioms.) Suppose further that each connected component of $R^2 - G$ is a simply connected region bounded by a Jordan curve. (This means that G is connected and cannot be disconnected by the removal of a single edge.) It is now clear that each such Jordan curve is made up of a finite number of vertices and edges; that each edge is adjacent to exactly two regions of $R^2 - G$, necessarily distinct, one on each side of the edge; and that the union of the closures of all these regions is R^2.

If all these conditions are satisfied, the regions of $R^2 - G$ are said to form a *map*, M(say), in the plane, and we talk about vertices, edges, and regions of M rather than of G or of $R^2 - G$. Two regions of M are *adjacent* if their boundaries have a common edge. A set in the plane is said to be *divisible into regions* if there is a map M in the plane such that the set consists of the union of some regions of M and possibly some other points all of which are in the boundaries of those regions. Several sets are said to be *simultaneously divisible into regions* if this statement is true with the same map M for each of the sets, and if each region of M belongs to at most one of the sets.

THEOREM 5. *If there is a covering of the plane by n closed sets $A_1, A_2, \ldots, A_n$ such that for each set A_i there is a distance d_i that is not realized as the distance between any two points of A_i, then there is a covering of the plane by n closed sets $B_1, B_2, \ldots, B_n$ that are simultaneously divisible into regions, such that B_i fails to realize the same distance d_i as A_i $(i = 1, 2, \ldots, n)$. Hence* C(a) $\geqslant$ R(a) *and* C(b) $\geqslant$ R(b).

Proof. Let S be any closed unit square of the infinite unit square lattice in the plane. Since each set A_i is closed and fails to realize distance d_i, there exist numbers $\epsilon_i(S)$ ($i = 1, 2, \ldots, n$) such that no two points of A_i, one or both of which are in S, realize any distance d such that

$$d_i - \epsilon_i(S) \leqslant d \leqslant d_i + \epsilon_i(S).$$

Let $\epsilon(S) := \min\{\epsilon_i(S) : i = 1, 2, \ldots, n\}$.

For each such square S, choose $n(S)$ such that

$$(\tfrac{1}{2})^{n(S)} \sqrt{2} < \tfrac{1}{2}\epsilon(S),$$

and consider the covering of S by $2^{2n(S)}$ closed squares of the lattice of side $(\frac{1}{2})^{n(S)}$, in the obvious way. For each such little square s in each unit square S in the entire plane, define $f(s) := \min\{i : s \cap A_i \neq \phi\}$, and then define

$$B_i := \bigcup_{f(s)=i} s \qquad (i = 1, 2, \ldots, n).$$

The B_i are closed and cover the plane, their interiors are disjoint, and each B_i fails to realize the corresponding distance d_i. If we make each corner of each square s a vertex, and each side an edge, we construct the required map M and the proof of Theorem 5 is complete. ∎

THEOREM 6. $\mathrm{R(b)} \geqslant 6$.

Proof. Suppose that the plane is covered by five sets, A, B, C, D and E, which are simultaneously divisible into regions with map M (say), in such a way that no two points that are unit distance apart are in the same set. We must derive a contradiction, which we shall do in a sequence of numbered steps. We start by progressively simplifying the covering, and to begin with we make no use of the fact that there are only five sets.

Step 1. We may suppose without loss of generality that no two adjacent regions of M are in the same set of the covering. For, if they are, the whole of the edge separating them is in that set (and in none other), and so we may delete that edge from M and merge the two regions. We may then delete any vertex that has become isolated in this way. If our new 'M' is not now a map, it can only be because some region or regions are completely contained within another region (i.e., the graph of the edges has become disconnected); and in this case we can put the interior region(s) into the same set as the surrounding region without making that set realize unit distance, and we can delete the intervening edges and vertices. So we shall suppose from now on that each edge separates regions in different sets.

Step 2. We may suppose, without loss of generality, that each edge enters each of its vertices in a well-defined direction, in a sense that we shall now make precise. It is convenient here to regard the plane as the complex plane.

Let P be one vertex of an edge e, and let I denote the closed unit interval. Since e is a Jordan arc, there is a continuous 1–1 correspondence $f: I \to e$ such that $f(0) = P$. If $x \in I$, let $\theta(x) := \arg(f(x) - P)$ $(-\pi < \theta(x) \leqslant \pi)$. The assertion is that we may assume, without loss of generality, that $\lim \theta(x)$ exists as $x \to 0$.

Suppose that $\lim \theta(x)$ does not exist. Then there is some interval $[\theta_1, \theta_2]$ such that, for each $\theta \in [\theta_1, \theta_2]$ and each $\epsilon > 0$, there exists $x < \epsilon$ such that $\theta(x) = \theta$. Choose such θ_1 and θ_2 satisfying $\theta_2 - \theta_1 < \frac{1}{2}\pi$. Let l_1, l_2, and l denote the (half-) lines emanating from P with arguments θ_1, θ_2, and $\frac{1}{2}(\theta_1 + \theta_2)$, respectively. Each of the two unit circles touching l at P intersects exactly one of the lines l_1 and l_2: let P_1 be the point where one of these circles meets l_1, and P_2 the point where the other meets l_2. By hypothesis we can find points Q_1 and Q_2 of e lying strictly within the segments PP_1 and PP_2 of l_1 and l_2, respectively. Since $\theta_2 - \theta_1 < \frac{1}{2}\pi$, there is one unit circle through Q_1 and Q_2 that does not include P in its interior and therefore intersects l twice. Let R be the point nearer P where this circle meets l, and Q a point of e strictly within the segment PR of l. We shall prove that we can modify e by substituting the segment PQ of l for the segment PQ of e.

Each unit circle through each point strictly within the segment PQ of l, except for the unit circles through that point and P, must separate Q_1 from P or Q_2 from P, and so must cross e. If X and Y are the two regions separated by e, it follows that each such unit circle passes through both X and Y, and consequently that its center is not in the same set of the covering as X or Y. Moreover, this last remark holds even for the unit circles through P: for the centers of these lie on a segment of the unit circle with center P, and a region on either side of this segment consists entirely of points that are the centers of other unit circles of the type already referred to; none of these neighboring points is in the same set as X or Y, and so none of the points of the segment can be, by the definition of "divisible into regions."

As predicted, we now modify the edge e by replacing the segment of e joining P to Q by the segment of straight line PQ. This has the effect of transferring some points from X to Y and some from Y to X. The points that are transferred in this way can be divided into regions, each of which is bounded by a Jordan curve consisting of a segment of e and a segment of PQ. Each unit circle through each point in each such region must thus intersect either e, and so pass through X and Y, or PQ. In either case the center of the circle cannot be in the same set as X or Y. So no point in any of these transferred regions is at unit distance from any point in the same set as X or Y, and transferring these regions from X to Y or from Y to X does not violate the conditions of the covering.

As a result of all this we shall suppose from now on, without loss of generality, that there is a well-defined direction in which an edge enters a vertex, and thus that it is meaningful to talk about the angle subtended by a region at a vertex. Clearly the sum of all the angles subtended at any vertex by the regions adjacent to it is 2π.

Step 3. We may suppose without loss of generality that, if there are k regions (say) adjacent to a vertex P, then at least $k - 1$ sets of the covering are represented among those k regions, and that if two of the regions are in the same set, then each of them subtends angle 0 at P. (Here we use the fact that there are only five sets in the covering.) For suppose two regions X and Y at P are in the same set. X and Y do not have an edge in common at P, in view of the supposition made in step 1, and so X and Y divide the other regions at P into two sectors, which are not adjacent at P. Suppose that one of these sectors subtends an angle less than π at P. Then there exists $\epsilon > 0$ such that each point Q in that sector, whose distance from P is less than ϵ, has this property: each unit circle through Q, except for those through P, passes through X or Y. As in step 2 above, we deduce that Q is not at unit distance from any point in the same set as X or Y. Thus we can add all such points Q to the same set as X and Y, making X and Y now part of the same region, without violating the conditions of the covering. In fact, we can do this unless X and Y are completely contained within the "cone" outside two unit discs touching each other at P. Moreover, in this exceptional case it is easy to see, by a similar argument, that we may extend X and Y if necessary so that they fill the whole of this "cone" within some distance ϵ of P.

Let us suppose that we have carried out all the extensions of the above form that are possible. Suppose that at some vertex P there are now k regions representing fewer than $k - 1$ sets. These k regions must include at least two "cones," belonging to different sets, and there must be at least four other sets represented between these cones: thus there are at least six covering sets altogether, and this is the contradiction we are seeking. So we shall suppose from now on that, if there are k regions adjacent to a vertex P, then either they are all in different sets and $3 \leqslant k \leqslant 5$; or there are $k - 1$ sets represented, two of the regions fill a "cone" of the type described in the neighborhood of P, and $3 \leqslant k - 1 \leqslant 5$.

Step 4. The next step is to prove what is effectively a lemma. But first we require a definition, for which we continue to regard the plane as the complex plane. A set R in the plane is said to be an *interior*, respectively *exterior*, *unit arc of positive thickness* of length ϕ with center P, if there is an open interval (θ_1, θ_2) with $\theta_2 - \theta_1 = \phi$, and a function f defined on (θ_1, θ_2) such that $f(\theta) < 1$, respectively $f(\theta) > 1$, for all θ $(\theta_1 < \theta < \theta_2)$, such that

$$R = \{z : \arg(z - P) =: \theta, \quad \theta_1 < \theta < \theta_2 \quad \text{and} \quad f(\theta) < |z - P| < 1\},$$

respectively,

$$R = \{z : \arg(z - P) =: \theta, \quad \theta_1 < \theta < \theta_2 \quad \text{and} \quad 1 < |z - P| < f(\theta)\}.$$

The lemma is that *an (interior or exterior) unit arc of positive thickness of length $\frac{2}{3}\pi$ cannot lie entirely in two of the covering sets.* The proof is fairly clear: the arc—in fact, any arc of length greater than $\frac{1}{3}\pi$—must be crossed by an edge e (say), which separates regions in two sets; the point, on the segment of unit circle bordering the unit arc, that is at $\frac{1}{3}\pi$ from the point where e crosses this segment, is in or on a band every point of which is at unit distance from points neighboring e on both sides; points of this band thus require a third set. There are two such bands, one on each side of e, at least one of which must cross the unit arc—even if e, where it crosses the arc, is itself in the form of a segment of unit circle centred on an end point of the arc where the arc has zero thickness. (The lower bound $\frac{2}{3}\pi$ is critical here.)

Step 5. We may suppose without loss of generality that each region subtends an angle of more than $\frac{1}{3}\pi$ at each of its vertices (and hence, in view of step 3 above, that there are no "cones" and that all the regions at any one vertex are in different sets). For suppose that some region subtends an angle $\phi \leqslant \frac{1}{3}\pi$ at a vertex P: let this region be in set B (say) and the neighboring regions, necessarily in different sets, be in sets A and C. Then opposite the region in B there is an interior unit arc of positive thickness of length $\pi - \phi \geqslant \frac{2}{3}\pi$ with center P, no point of which can be in A, B, or C, and every point of which is therefore in D or E. And this we know is impossible by step 4 above. So we shall suppose from now on that every angle subtended is greater than $\frac{1}{3}\pi$.

Step 6. Suppose that P is a vertex with exactly three neighboring regions, in sets A, B, and C (say). P is surrounded by a complete unit annulus of positive thickness with center P, no point of which can be in A, B, or C, unless an edge incident with P is in the form of an arc of a unit circle. In this exceptional case there may be up to three points round the annulus where its thickness falls to zero. But between two of these points we can certainly find an arc of positive thickness of length at least $\frac{2}{3}\pi$ that is covered entirely by sets D and E, and this is impossible as before. (In fact, this "arc of positive thickness" is likely to be a union of interior and exterior arcs.) So we shall suppose from now on that each vertex is incident with at least four regions.

Step 7. Suppose that P is a vertex with exactly four neighboring regions, a, b, c, and d (say), in sets A, B, C, and D, respectively, in that order round P. Let region x subtend angle ϕ_x at P, and let ϕ_b be the smallest ϕ_x ($x = a$, b, c or d), where (in view of step 5 above) $\frac{1}{3}\pi < \phi_b \leqslant \frac{1}{2}\pi$. There is an interior unit arc of positive thickness of length $\pi - \phi_b \geqslant \frac{1}{2}\pi$, opposite b, which lies entirely in D and E. This must be crossed by an edge e, which separates regions in D and E. By the argument of step 4 above, e must terminate (which it can do only by meeting a vertex on the unit circle with center P) in the portion of the unit arc that is within $\frac{1}{3}\pi$ of both the points on this unit circle at the two ends of the arc. In the neighborhood of this vertex outside the interior arc the only sets which can be represented are B and E. (This is because $\phi_a > \frac{1}{3}\pi$ and $\phi_c > \frac{1}{3}\pi$: a diagram will make this clear. The strictness of these inequalities is necessary.) Thus this vertex has only three neighboring regions (in sets B, D, and E), which contradicts the supposition of step 6 above. So we shall suppose from now on that each vertex is incident with exactly five regions.

Step 8. This final step is similar to the preceding one. Suppose that P is a vertex with exactly five neighboring regions, a, b, c, d, and e (say), in sets A, B, C, D, and E, respectively, in that order round P. Let ϕ_b be the smallest ϕ_x ($x = a, b, c, d$, or e, with ϕ_x defined as in step 7 above), where $\frac{1}{3}\pi < \phi_b \leqslant \frac{2}{5}\pi$. There is an interior unit arc of positive thickness of length $\pi - \phi_b \geqslant \frac{3}{5}\pi$, opposite b, which lies entirely in D and E. For exactly the same reasons as before, this arc must be crossed by an edge, which separates regions in D and E and meets a vertex, on the unit circle bounding the unit arc, at which only sets B, D, and E can be represented. Thus again we have obtained a contradiction by exhibiting a vertex with only three neighboring regions.

The only remaining possibility is that there are no vertices at all, and hence no edges. But this is clearly impossible, since it would violate the conditions of the covering. So we conclude that the initial hypothesis of the proof, that there are only five sets, must be false. Thus $\mathrm{R(b)} \geqslant 6$. This completes the proof of Theorem 6. ∎

We may note in conclusion that, if the covering is by *closed* sets that are simultaneously divisible into regions with map M (say), then we can go considerably beyond this theorem. In particular, if we have a covering by six such sets none of which realizes unit distance, then the regions adjacent to each vertex of the map M represent at most three sets, and we may suppose without loss of generality that there are exactly three regions at each vertex, representing different sets. Moreover, it is clear that we cannot have two vertices, within distance 2 of each other, one of which has adjacent regions in three of the sets, and the other of which has adjacent regions in the other three sets. In fact it seems likely, although I do not claim to have proved this, that only six of the $\binom{6}{3} = 20$ possible triples of sets can occur as the sets represented at a vertex of M, and that these are of the form ABC and ABD, CDE and CDF, and EFA and EFB. But even if this is true, I do not see how to use the fact to prove that such a covering by six closed sets is impossible, nor how to construct such a covering.

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